FAMILIES OF VARIETIES OF GENERAL TYPE OVER COMPACT BASES

STEFAN KEBEKUS AND SÁNDOR J. KOVÁCS

1. INTRODUCTION

Let $f : X \to Y$ be a smooth family of canonically polarized complex varieties over a smooth base. Generalizing the classical Shafarevich hyperbolicity conjecture, Viehweg conjectured that Y is necessarily of log general type if the family has maximal variation. We refer to [KK05] for a precise formulation, for background and for details about these notions. A somewhat stronger and more precise version of Viehweg's conjecture was shown in [KK05] in the case where Y is a quasi-projective surface. Assuming that the minimal model program holds, this very short paper proves the same result for projective base manifolds Y of arbitrary dimension.

We recall the two relevant standard conjectures of higher dimensional algebraic geometry first.

Conjecture 1.1 (Minimal Model Program and Abundance for $\kappa = 0$)**.** *Let Y be a smooth projective variety such that $\kappa(Y) = 0$. Then there exists a birational map $\lambda : Y \dashrightarrow Y_\lambda$ such that the following holds.*

(1.1.1) *Y_λ is $\mathbb{Q}$-factorial and has at worst terminal singularities.*
(1.1.2) *There exists a number n such that nK_{Y_λ} is trivial, i.e., $\mathscr{O}_{Y_\lambda}(nK_{Y_\lambda}) = \mathscr{O}_{Y_\lambda}$*

Conjecture 1.2 (Abundance for $\kappa = -\infty$)**.** *Let Y be a smooth projective variety. If $\kappa(Y) = -\infty$, then Y is uniruled.*

Remark 1.3. Conjectures 1.1 and 1.2 are known to hold for all varieties of dimension $\dim Y \le 3$.

The main result of this paper is now the following, cf. [KK05, Conjecture 1.6].

Theorem 1.4. *Let Y be a smooth projective variety and $f : X \to Y$ a smooth family of canonically polarized varieties. Assume that Conjectures 1.1 and 1.2 hold for all varieties F of dimension $\dim F \le \dim Y$. Then the following holds.*

(1.4.1) *If $\kappa(Y) = -\infty$, then $\mathrm{Var}(f) < \dim Y$.*
(1.4.2) *If $\kappa(Y) \ge 0$, then $\mathrm{Var}(f) \le \kappa(Y)$.*

Remark 1.5. The argumentation of Section 2 actually shows a slightly stronger result. If $\kappa(Y) = -\infty$, it suffices to assume that Conjecture 1.2 holds for Y. If $\kappa(Y) \ge 0$, we need to assume that Conjecture 1.1 holds for all varieties F of dimension $\dim F = \dim Y - \kappa(Y)$.

See Theorem 3.1 below for further generalizations.

Theorem 1.4 and Remark 1.3 immediately imply the following.

Corollary 1.6. *Viehweg's conjecture holds for smooth families of canonically polarized varieties over projective base manifolds of dimension ≤ 3.* □

Date: February 1, 2008.

Stefan Kebekus was supported in part by the DFG-Forschergruppe "Classification of Algebraic Surfaces and Compact Complex Manifolds". Sándor Kovács was supported in part by NSF Grant DMS-0554697 and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics.

2. Proof of Theorem 1.4

2.A. **The case where $\kappa(Y) = -\infty$.** The assertion follows immediately from Conjecture 1.2 and from the fact that families of canonically polarized varieties over rational curves are necessarily isotrivial [Kov96, Thm. 1].

2.B. **The case where $\kappa(Y) = 0$.** In this case, we need to show that the family f is isotrivial. We argue by contradiction and assume that $\operatorname{Var}(f) \geq 1$. By [VZ02, Thm. 1.4.i], this implies that there exists a number n and an invertible subsheaf $\mathscr{A} \subset \operatorname{Sym}^n \Omega^1_Y$ of Kodaira-Iitaka dimension $\kappa(\mathscr{A}) \geq \operatorname{Var}(f) \geq 1$.

By assumption, there exists a birational map $\lambda : Y \dashrightarrow Y_\lambda$ as discussed in Conjecture 1.1. Resolving the indeterminacies of λ and pulling back the family f, we may assume without loss of generality that λ is a morphism, i.e., defined everywhere.

Let $C_\lambda \subset Y_\lambda$ be a general complete intersection curve. Then C_λ will avoid the singularities of Y_λ. In particular, the restriction $\Omega^1_{Y_\lambda}|_{C_\lambda}$ is a vector bundle of degree

$$\deg \Omega^1_{Y_\lambda}|_{C_\lambda} = K_{Y_\lambda} \cdot C_\lambda = 0. \tag{2.B.1}$$

Claim 2.1. The vector bundle $\Omega^1_{Y_\lambda}|_{C_\lambda}$ is not semi-stable.

Proof of Claim 2.1. Observe that the curve C_λ avoids the fundamental points of λ, and hence that λ is an isomorphism in a neighborhood of C_λ. Setting $C := \lambda^{-1}(C_\lambda)$, the morphism λ induces an isomorphism $\Omega^1_{Y_\lambda}|_{C_\lambda} \cong \Omega^1_Y|_C$. This shows that $\Omega^1_{Y_\lambda}|_{C_\lambda}$ cannot be semi-stable, for if it was, its symmetric product $\operatorname{Sym}^n \Omega^1_{Y_\lambda}|_{C_\lambda}$ would also be semistable of degree 0. However, this contradicts the existence of the subsheaf $\mathscr{A}$ whose restriction to C has positive degree. $\square$

To end the proof, observe that (2.B.1) and Claim 2.1 together imply that $\Omega^1_{Y_\lambda}|_{C_\lambda}$ has an invertible quotient of negative degree. In this setup, Miyaoka's uniruledness criterion, cf. [Miy87, Cor. 8.6], [KST07] or [KSC06, Chapt. 2.1], applies to show that Y is uniruled, contradicting the assumption that $\kappa(Y) = 0$.

2.C. **The case where $\kappa(Y) > 0$.** In this case, consider the Iitaka fibration of Y, $\mathfrak{i} : Y' \to Z$. Since the Iitaka model is only determined birationally, we may assume that there exists a birational morphism $Y' \to Y$. Pulling the family $f : X \to Y$ back to Y', we may assume that $Y' = Y$, and hence we may assume that there exists a fibration $\mathfrak{i} : Y \to Z$ such that $\dim Z = \kappa(Y)$ and $\kappa(F) = 0$ for the general fiber F of $\mathfrak{i}$ [Iit82, Thm. 11.8]. We have seen in Section 2.B that $f|_F$ is isotrivial and hence $\operatorname{Var}(f) \leq \dim Y - \dim F = \dim Z = \kappa(Y)$. This finishes the proof of Theorem 1.4.

3. Families of varieties of general type

Using [VZ02, Thm. 1.4.iii], the argumentation of Section 2 immediately gives the following, somewhat weaker, result for families of varieties of general type.

Theorem 3.1. *Let Y be a smooth projective variety and $f : X \to Y$ a smooth family of varieties of general type of maximal variation, i.e.,* $\operatorname{Var}(f) = \dim Y$*. If Conjectures 1.1 and 1.2 hold for all varieties F of dimension* $\dim F \leq \dim Y$*, then Y is of general type.* $\square$

References

[Iit82] S. Iitaka: *Algebraic geometry*, Graduate Texts in Mathematics, vol. 76, Springer-Verlag, New York, 1982, An introduction to birational geometry of algebraic varieties, North-Holland Mathematical Library, 24. MR637060 (84j:14001)

[KK05] S. Kebekus and S. J. Kovács: *Families of canonically polarized varieties over surfaces*, preprint math.AG/0511378, to appear in Invent. Math., November 2005.

[KSC06] S. KEBEKUS AND L. SOLÁ CONDE: *Existence of rational curves on algebraic varieties, minimal rational tangents, and applications*, Global aspects of complex geometry, Springer, Berlin, 2006, pp. 359–416. MR2264116

[KST07] S. KEBEKUS, L. SOLÁ CONDE, AND M. TOMA: *Rationally connected foliations after Bogomolov and McQuillan*, J. Algebraic Geom. **16** (2007), no. 1, 65–81.

[Kov96] S. J. KOVÁCS: *Smooth families over rational and elliptic curves*, J. Algebraic Geom. **5** (1996), no. 2, 369–385. MR1374712 (97c:14035)

[Miy87] Y. MIYAOKA: *Deformations of a morphism along a foliation and applications*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 245–268. MR927960 (89e:14011)

[VZ02] E. VIEHWEG AND K. ZUO: *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328. MR1922109 (2003h:14019)

STEFAN KEBEKUS, MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, WEYERTAL 86–90, 50931 KÖLN, GERMANY

E-mail address: stefan.kebekus@math.uni-koeln.de

URL: http://www.mi.uni-koeln.de/∼kebekus

SÁNDOR KOVÁCS, UNIVERSITY OF WASHINGTON, DEPARTMENT OF MATHEMATICS, BOX 354350, SEATTLE, WA 98195, U.S.A.

E-mail address: kovacs@math.washington.edu

URL: http://www.math.washington.edu/∼kovacs